# A Gaskinetic Model for Planar Plume and Comparisons with the Simons Plume Model

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***Abstract***

This paper presents some validation results for a set of recently developed gaskinetic solutions to rarefied free planar jets expanding into a vacuum. The traditional semi-empirical cosine law or the Simons rocket plume model and the direct simulation Monte Carlo (DSMC) numerical results are used to compare this set of analytical solutions. For the analytical and DSMC results, there is excellent agreement on flows at large Knudsen (Kn) numbers; the new model also provides more accurate density results at low Knudsen number than the traditional Simons model. The new analytical solutions further include velocity, temperature, and pressure formulas. This new model can serve as a baseline reference to evaluate, design and optimize high speed plume/jet flows.

***Keywords***

*Jet; Plumes; DSMC Method; Propulsion; Rocket Plume; Kinetic Theory; Collisionless; Nonequilibrium Flows*

## Introduction

Gaseous plumes have many important applications, such as rocket performance evaluation and rocket target detection, materials processing inside vacuum chambers, micro-propulsion, and molecular beams. The flow fields generated by rocket propulsion systems are complex and include regions of strong inviscid/viscous interactions, free-stream shear layers, separation region sand downstream location plume/plume interactions. Plume models can be characterized as either detailed or analytical ones. The former are based on patently theoretical treatments, and the latter are simple representations based on more approximate treatments of the complex gas dynamic and radiative processes.

There are numerous reports in the literature, on experimental and numerical studies of nozzle flows. However, studies using analytical methods are rare. Most analytical studies concentrate on fluxes inside a duct or at nozzle exit, for example, the work by Lipemann. Usually people adopt two types of approaches, one from the continuum flow side or the collisionless side. For the latter one, some simplifications are adopted and many of them only provide density results.

Here we would like to provide two examples. One is the popular cosine law or the Simons model. It expresses the plume density in terms of the boundary-layer thickness and nozzle exit conditions. It is a continuum, semi-empirical, far-field approximation solution which is widely used for rocket plume research. Many researchers chose the Simons model to analyze rocket plume flows. For example, Manzella and Carney used it in the investigation of liquid-fed water resisto-jet plume; Kannenberg and Boyd used this model to compute the density of a plume impinging at a plate. Another example is the work by Narasimha, who obtained the exact solutions of density and velocity distributions for a free molecular effusion flow with a zero average macroscopic speed out of a circular slit. For the case of free molecular flows out of a nozzle with a nonzero average velocity, Narasimha's investigation indicated that the solution for the plume is rather complicated with many cosine functions, and only provided simplified density results.

For rarefied nozzle/plume flows, there are three adopted nozzle shapes: planar, circular/annular and rectangular. Recently, Cai and Boyd provided two sets of detailed macroscopic solutions for collisionless planar, circular or planar plume flows. These solutions are further extended the solutions to 2D/3D plume impingement on a flat plate, which can be used to estimate rocket plume impingement on spacecraft solar panels. As the first order estimations on the rarefied plume flows, these new solutions only consider collisionless flow situations, such as those fired from an electric propulsion device. The collision effects are completely neglected.

This paper is a continued effort from another paper on numerical validations of these recently developed

collisionless jet flow models and solutions. This paper has two purposes. Firstly, it examines the performance of these new solutions for the planar jet/plumes; which is necessary to provide detailed validations to demonstrate the application scope and emphasize the assumptions for these solutions. Secondly, by comparing these solutions with the traditional cosine law or the Simons plume model, it can be illustrated that the gaskinetic solutions are more comprehensive with several advantages over those past continuum plume models.

This paper is organized as follows: Section II presents the analytical solutions and the cosine law or the Simons plume model; Section III provides validation results by comparing the results among analytical, DSMC and the Simons models; and Section IV summarizes this paper with a few conclusions.

## Collisionless Analytical and Empirical Rocket Plume Models

### *Collisionless Planar Free Jet Expanding into Vacuum*

The problem of a collisionless planar free plume expanding into vacuum is illustrated by FIG. 1. The jet starts from a planar exit at X=0. A previous study proposed that if we assume at the exit, the gas is at equilibrium state which is characterized by a number density $n_0$, an average velocity $U_0$, and a temperature $T_0$, then ata flowfield point P(X, Y), the exact gaskinetic collisionless solutions of number density, velocity and temperature are:

$$n_1(X,Y) = \frac{\exp(-s_0^2)}{2\pi}(\theta_2-\theta_1) + \frac{1}{4}[\mathrm{erf}(S_0\sin\theta_2) - \mathrm{erf}(S_0\sin\theta_1)] + \frac{S_0}{2\sqrt{\pi}}\int_{\theta_1}^{\theta_2}\exp(-S_0^2\sin^2\theta)\cos\theta\,\mathrm{erf}(S_0\cos\theta)\,d\theta \quad \mathbf{(1)}$$

$$\frac{U_1(X,Y)}{\sqrt{2RT_0}} = \frac{\exp(-s_0^2)}{2\pi n_1}\Big(\int_{\theta_1}^{\theta_2}\{\frac{\sqrt{\pi}}{2}\exp(S_0^2\cos^2\theta)\cos\theta[1+\mathrm{erf}(S_0\cos\theta)]\}\,d\theta + \frac{S_0(\theta_2-\theta_1)}{2} + \frac{S_0}{4}[\sin(2\theta_2)-\sin(2\theta_1)] + S_0^2\sqrt{\pi}\int_{\theta_1}^{\theta_2}\cos^3\theta[1+\mathrm{erf}(S_0\cos\theta)]\exp(S_0^2\cos^2\theta)\,d\theta \quad \mathbf{(2)}$$

$$\frac{V_1(X,Y)}{\sqrt{2RT_0}} = \frac{1}{4n_1\sqrt{\pi}}\{\exp(-S_0^2\sin^2\theta_1)\cos\theta_1[1+\mathrm{erf}(S_0\cos\theta_1)] - \exp(-S_0^2\sin^2\theta_2)\cos\theta_2[1+\mathrm{erf}(\cos\theta_2)]\} \quad \mathbf{(3)}$$

$$T_1(X,Y) = -\frac{U_1^2+V_1^2}{3R} + \frac{T_0\exp(-S_0^2)}{6n_1\pi}\{3(\theta_2-\theta_1) + S_0^2[\theta_2-\theta_1+\frac{1}{2}(\sin(2\theta_2)-\sin(2\theta_1))] + 2S_0\sqrt{\pi}\int_{\theta_1}^{\theta_2}(2\cos\theta+S_0^2\cos^3\theta)[1+\mathrm{erf}(S_0\cos\theta)]\exp(S_0^2\cos^2\theta)\,d\theta \quad \mathbf{(4)}$$

where $S_0 = U_0/\sqrt{2RT_0}$ is the characterized exit speed ratio. The above results are valid in the entire domain, not only at farfield.

### *The Cosine Law or the Simons Plume Model*

For rocket plumes, there are many different plume models. The model we chose to compare with the gaskinetic solutions is the concise cosine law or the Simons model. This model includes an expression for density which makes it convenient for implementation. Far from the rocket, rocket nozzle flows may be treated as though it originates from a point source. Using the continuity equation, the density at any point in the plume can be calculated in a closed form based on the condition in the nozzle with the following equation:

$$\frac{\varrho(r,\theta)}{\varrho_s} = A\left(\frac{R'}{r}\right)^2 f(\theta) = A\left(\frac{R'}{r}\right)^2\cos^{\kappa}\left(\frac{\sqrt{\pi}}{2}\frac{\theta}{\theta_{max}}\right) \quad \mathbf{(5)}$$

Where **A** is a constant obtained from the continuity of the rocket-mass flow, and is a function of the specific heat ratio and exit Mach number. The *f(θ)* function is a plume angular density decay function. The choice of this function for inviscid flows has received considerable attention. Boyton's choice of f(θ) is constrained to obey the locally 2D Prandtl-Meyer function for θ near $\theta_{max}$, and this constraint together with correlations of numerical results, indicates that f(θ) is best duplicated by a cosine power function where **R'** is the orifice radius, $\theta_{max}$ is an angular scale, and κ is a constant determined by gas specific heat ratio. It is an empirical value used to account for observed deviation at non-zero average velocity of the plume. This Simons model is based on Knudsen cosine law behaviour for free molecular effusion at zero average velocity. The exponent κ frequently referred to as "beaming" exponent is the empirical method to account for observed deviation at non-zero average velocity of the beam/plume. For example, Ashkenas used κ=2,Albini used κ = 1/(γ-1) while Boyton suggested κ= 2/(γ-1).

The cosine law or the Simons plume model may actually not fit physics well, because the mass flux integrals are not constant across different angular lines of different radius, $R_1$ and $R_2$.That is because according to the gaskinetic theory, along a fixed θ line, $V_r(\theta)$

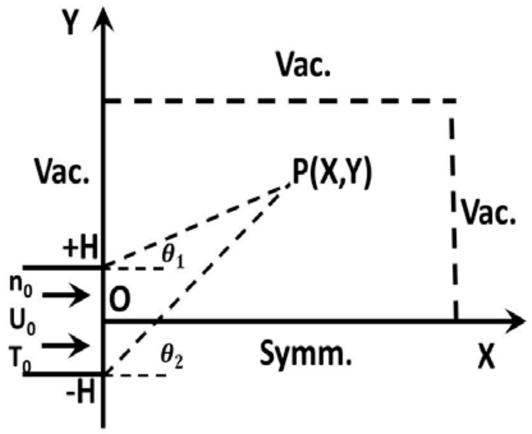


FIG.1 ILLUSTRATION OF THE PROBLEM, THE SIMULATION DOMAIN AND BOUNDARY CONDITIONS

continues to increase because higher speed molecules have larger opportunity to be observed or counted. For collisionless flows there is no mechanism to alter molecules' velocity directions and magnitudes, this fact is especially correct. We can understand that for high Knudsen number flows, such as the plasma plume flows from electric propulsion devices, the Simons model does not perform well. Most likely this is the reason that people use different κ values in their work to obtain an optimal match of their simulation results with different plume exit speed ratio $S_0$ values. It is evident that different plume exit average speed ratio shall contribute to the plume structure; however, this factor is neglected in the Simons model. This is a conflict with the physical fact a supersonic plume shall have a narrower plume core than a subsonic case because molecules have shorter time to diffuse vertically.

Do we have better options than the cosine law or the Simons plume model? Our answer is the newly developed gaskinetic model may be such a candidate. The density profiles $n_1(X, Y)$ involve the average plume exit speed ratio $S_0$, and functions much complex than single $cos^{\kappa}\theta$. As such, we can expect the recently developed gaskinetic model to offer improved performance.

At farfield, due to the existence of gaskinetic velocity asymptote, we can expect that mass flow rate, is almost constant at locations far from the nozzle. As such, the Simons model may be able to provide better predictions there.

## Validations

A planar free jet expanding into vacuum at different Knudsen numbers is simulated with the DSMC method, and compared with those analytical formulas, and the Simons model. For the DSMC simulations, the test gas is Argon and we assume the nozzle semi-height H = 0.5 m. We adopted a special DSMC package GRASP for these simulations. As illustrated by FIG. 1, an inlet boundary is used to represent the nozzle, a symmetric line for the centerline, and vacuum boundaries for other sides. Here we use a special representative $S_0$=2 for our validation. Collisions of molecules are simulated using the Variable Hard Sphere (VHS) model for less rarefied flows, and no time counting (NTC) method is adopted for the simulations.

FIG. 2 compares normalized number density contours for Knudsen number 100. It is evident that the analytical and DSMC results are essentially identical as expected.

FIG. 3 shows the normalized U-velocity component contours at Kn = 100. The analytical and DSMC results agree perfectly for this case. Along the plume centreline, the velocity is high, because most particles are carried down quickly with the high exit speed. At the outer region of the plume are those slow particles with lower U-Velocity components; they have longer time to travel downstream-wisely. So they can diffuse vertically towards the plume outer boundary. As a result slower layers are formed; they shall be a small portion, so the density there cannot be large. This corresponds to the density contours in FIG. 2, FIG. 2 and 3 actually provide an explanation for the so called boundary layer, from a gaskinetic point of view.

FIG 4 shows the normalized V-velocity component contours at Kn = 100. The analytical and DSMC results agree well for this case. Along the plume centreline, the velocity is 0 due to the symmetric conditions. Along the Y-axis, the velocity is zero again, because there is no molecules travel upwards without any collisions. Between these two limiting lines, we have positive velocity particles, because molecules can only diffuse upward without any collisions. As results, the simplest profile for $V_y$ is to start from zero, increase, decrease, and drops to zero. These patterns are exactly reflected by FIG. 4. At the top region of FIG. 4, the DSMC simulation results have larger statistical scatters, because the density there is very low. At the exit, the larger statistical scatter is due to the almost zero V-velocity components.

From these two velocity component figures, we can observe that as r increases but fixed θ, $V_r$ increases. With a certain range of lower Kn numbers, we expect

this pattern to hold. This indicates the Simons model will not have good performance in those regions.

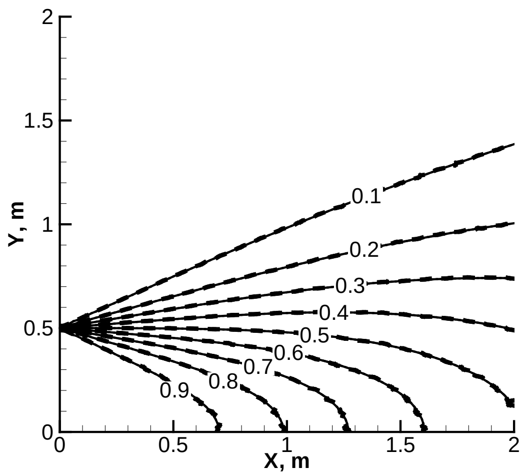


FIG.2 CONTOURS OF NORMALIZED NUMBER DENSITY. SOLID: ANALYTICAL, DASHED: DSMC, KN =100, H=0.5, $S_0$=2.0

FIG. 5 shows the pressure contours. The topology is the same as the density. FIG. 6 presents two number density contours with Kn=1 and Kn=0.01. As we compare with FIG. 2 where the analytical and DSMC results at Kn=100, we can observe that there is not much difference among these 3 pictures. The recently developed gaskinetic results are not only the exact solutions for collisionless plume flows, but also they can be used to approximate less rarefied jet/plume flows.

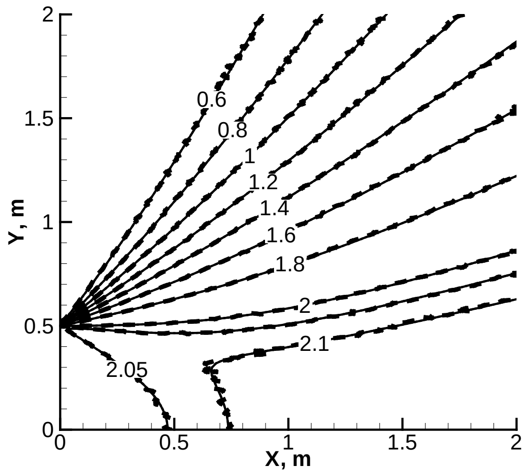


FIG. 3 CONTOURS OF NORMALIZED U- VELOCITY COMPONENT. SOLID: ANALYTICAL, DASHED: DSMC, KN=100; H=0.5, $S_0$=2.0

Also, it is worthy to mention that in these above 6 pictures, the nozzle exit lip is a singularity point due to the large gradients there.

The next two figures aim to address the following two questions regarding the density profiles: the Kn number range that we can apply the above gaskinetic model, and the performance against the cosine law or the Simons model. FIG. 7 shows the centreline density profiles obtained from analytical formula, finite Kn number DSMC simulation, and the Simons model. It is evident that along the centerline, the analytical and the DSMC results agree well, and all start from 1. As can be seen, even the gaskinetic density formula is based on the assumption of collisionless flow, however, the result can be used to estimate supersonic continuum flows as well even at Kn=0.001, which is a near-continuum flow situation. This is because with a high average supersonic mean flow velocity, most molecules have less time to collide and diffuse vertically. This holds even at continuum flows. Due to this very reason, even for continuum flows, as long as the average velocity is supersonic, they can be treated as collisionless as well. This is also reflected by relation Kn~Ma/Re. By comparison, the Simons model tends to have a very large value at the exit because it is a singularity point for this model. The Simons model treats the far field rocket nozzle flow as it originates from a point source and it intends to be applied for the farfield only. FIG. 7 plots the trends of the Simons model; it has different curvature, opposite to the DSMC simulation results. We have enough evidence to conclude that the cosine law or the Simons model provides poor performance, at least for the large Kn number scenarios, as well as at the near exit region.

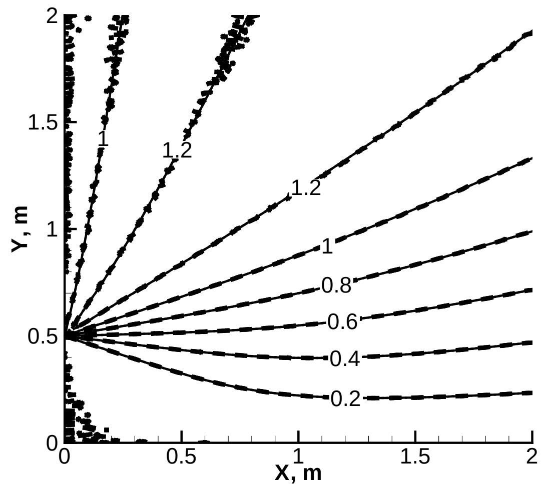


FIG. 4 CONTOURS OF NORMALIZED V-VELOCITY COMPONENT. SOLID: ANALYTICAL, DASHED: DSMC, KN=100; H=0.5, $S_0$=2.0

FIG. 8 shows comparison of density profiles along a vertical line of X=2m: the analytical collisionless solution, the DSMC simulation results, and the normalized Simons model (by the centerline value). Near the centreline, the DSMC simulation results and

the analytical results have some differences. This is understandable due to more collisions at lower Kn numbers, but for the large outer plume region, the agreement is very good. By comparison, even the Simons model has the same trends with different κ, different choices of κ yield divergent patterns of density profiles. Further, the starting value for the Simons model at Y=0 is not valid, as reflected in FIG.7.

From the above results, we conclude that the analytical results are not only identical to the DSMC simulation results for the collisionless flows, but also represent near continuum small Kn number plume flows. The Simons model's results are much inferior, and are only applicable at locations far from the exit. In practical applications, other properties such as pressure and velocity may be important but the cosine law or the Simons model could not offer any suggestions for these solutions.

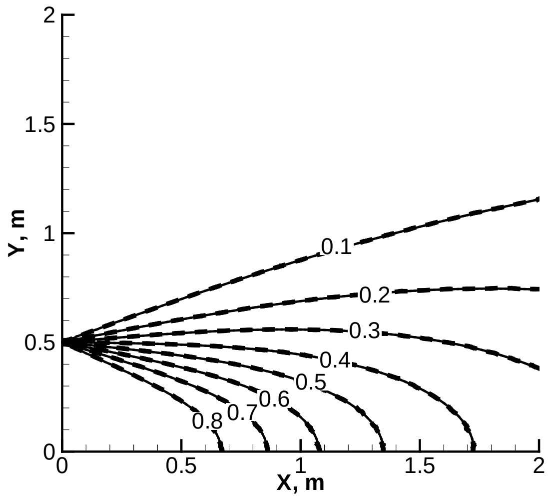


FIG.5 CONTOURS OF NORMALIZED PRESSURE COMPONENT. SOLID: ANALYTICAL, DASHED: DSMC, KN=100, H=0.5, $S_0$=2.0

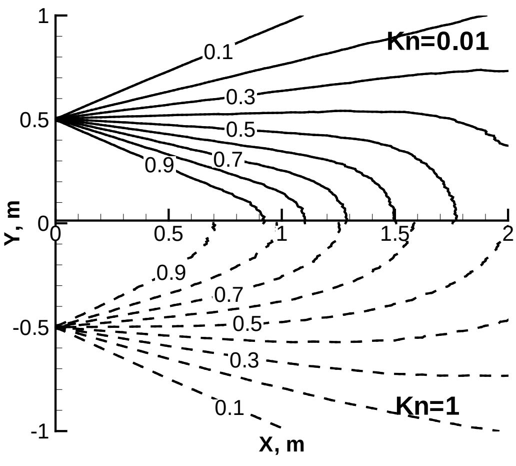


FIG.6 COMPARISONS OF DENSITY FIELD AT DIFFERENT KN NUMBERS

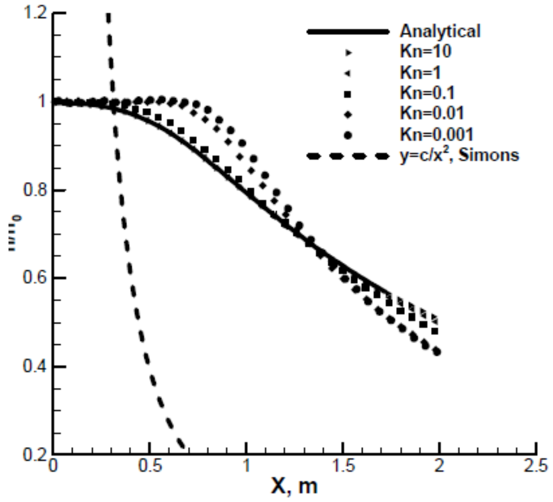


FIG.7 CENTRELINE DENSITY PROFILES, ANALYTICAL, DSMC, AND THE SIMONS MODEL

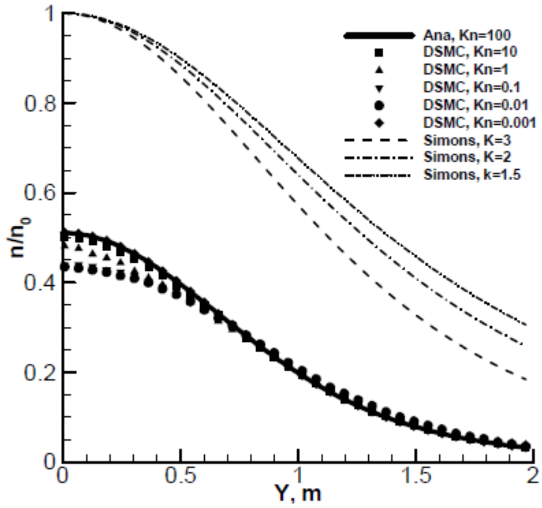


FIG.8 DENSITY PROFILES ALONG X=2 M: ANALYTICAL, DSMC, AND THE SIMONS MODEL

## Conclusions

In this paper, we briefly reviewed a recently developed gaskinetic model and the results by Cai for planar jet/plume flows. Then we presented several DSMC simulations of rarefied planar rocket plumes with different Kn numbers, where the characteristic length is the exit width. The DSMC results are further compared with the Simons model and the new gaskinetic rocket plume model.

The Simons model is concise and has been widely used by people for at least half a century to estimate plume density. However the agreement with the DSMC simulation results is poor, it is for farfield only, point source, semi-experimental, developed for continuum flows. For the case of higher Kn number, the analytical plume properties are virtually identical with the DSMC results; at lower Kn numbers, even at the near

continuum ranges, the gaskinetic model provides relatively better agreement with the DSMC simulation results.

From these comparisons, we can conclude that the detailed, complete rarefied plume solutions are more comprehensive and accurate than the Simons model, over the full Kn number range, including the near continuum flows, and a full flowfield. They also provide extra information such as velocity, temperature and pressure. Meanwhile, these new solutions are faster to compute than DSMC simulations, without any statistical scatters. As such, we recommend them to compute rocket plume fields, for comprehensive evaluations, designs and optimizations.

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